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Crack-tip field for fast fracture of an elastic–plastic–viscoplastic material coupled with quasi-brittle damage. Part 2. Small damage regime

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Abstract

A parameter perturbation technique is used to obtain asymptotic solutions that apply to a fast moving crack-tip, where small damage condition prevails. The material can be described by an elastic–plastic–viscoplastic constitutive relation including quasi-brittle damage. A dimensionless coefficient, which shows the characteristic damage within this regime, is taken as a perturbation parameter. A set of asymptotic equations is derived in terms of a regular perturbation expansion procedure. Asymptotic solutions are obtained for radial and angular variations of stresses and velocities with first- and second-order accuracy. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The present paper investigates the local solution of the stress field of fast fracture in a regime with small damage around the crack tip. Unlike the situation in the zone (Region A of Fig. 1) where the damage effect plays a dominant role such that singular stresses do not occur at the crack tip (Lu et al., 2001a), damage within Region C (of Fig. 1) remains at a relatively low level and stress concentration controls the stress field. In fact, in the vicinity of the crack tip, the stress concentration due to the presence of a crack and the relaxation of stresses because of damage are two competing mechanisms, and they play distinct roles in different regions. At the moving crack tip the damage reaches its critical state and complete failure is

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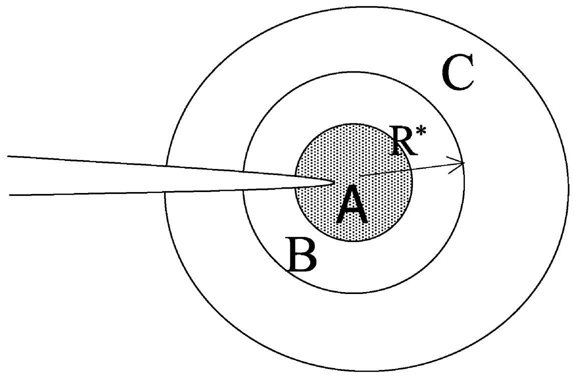

Fig. 1. Schematic illustration of stress fields around the crack tip.

achieved at the corresponding material point. Correspondingly, stresses at the crack tip tend to vanish since a fully damaged meso-element of material cannot sustain loads. With increasing distance away from the tip, the influence of damage gradually decreases whilst that of stress increases. Eventually, stress relaxation due to damage is suppressed by stress concentration and the influence of damage becomes a perturbation. When the perturbation of damage to the stress field vanishes completely, the analysis encounters the limiting case to which conventional fracture mechanics applies. (For example, see works conducted by Achenbach and Kanninen (1978), Achenbach et al. (1981), Amazigo and Hutchinson (1977), Freund and Douglus (1982), Leighton et al. (1987), Bose and Castañeda (1992), Östlund and Gudmandson (1988), Gao and Nemat-Nasser (1983), Gao et al. (1983) and Gao (1986).)

Since damage does not play a dominant role in the region concerned, the method used in Part 1 is no longer available. Alternatively, a regular parameter perturbation technique is used to deal with the behaviour in the regime where damage behaves as a perturbation. Through a dimensionless parameter, the regime where the small characteristic damage condition exists can be defined. This technique was employed to study the problem of a static crack in a damaged creeping body (Lee et al., 1997), and now it is extended to the current case in which the inertia effect and a more complex constitutive law are involved. It is expected that from the work presented in this Part, together with results given in Part 1, the behaviour of the crack-tip field can be better characterised from a different approach.

As in Part 1, the virgin material is still the elastic–plastic–viscoplastic material. Damage effects are incorporated into the constitutive relation by virtue of the strain-equivalence theorem of damage mechanics (Lemaitre, 1992). The kinetic evolution equation of damage is described using a quasi-brittle damage model.

All the basic equations and boundary conditions are given in Section 2. In Section 3 we normalise these equations and establish a small dimensionless quantity that defines the small damage condition. Then we use the regular parameter perturbation method to perform the asymptotic analysis. In Section 4 numerical computations for the angular variations of stresses and velocities are carried out, and in Section 5 concluding remarks are given.

2. Physical models and governing equations

2.1. Equation of motion

Basic governing equations are the same as those in Part 1 and they are listed below.

The Cartesian (x_1, x_2, x_3)-coordinate system is established with the x_3-axis lying along the crack front together with a polar coordinate system (R, θ). The origins of the coordinate systems move with the crack tip at velocity $\dot{a}(t)$ in the positive x_1-direction, where t is time. We consider a two-dimensional plane stress problem. The non-zero stress and displacement components are labelled by $\sigma_{11}(x_1, x_2, t)$, $\sigma_{22}(x_1, x_2, t)$, $\sigma_{12}(x_1, x_2, t)$ $(= \sigma_{21}(x_1, x_2, t))$, $u_1(x_1, x_2, t)$, and $u_2(x_1, x_2, t)$ in the Cartesian coordinate system or, equivalently, by $\sigma_{11}(R, \theta, t)$, $\sigma_{22}(R, \theta, t)$, $\sigma_{12}(r, \theta, t)$ $(= \sigma_{21}(R, \theta, t))$, $u_1(R, \theta, t)$, and $u_2(R, \theta, t)$ in the polar coordinate system.

The spatial derivatives in Cartesian and polar coordinate systems are related by

$$\frac{\partial}{\partial x_1} = \cos\theta \frac{\partial}{\partial R} - \frac{\sin\theta}{R} \frac{\partial}{\partial \theta} \tag{1}$$

and

$$\frac{\partial}{\partial x_2} = \sin\theta \frac{\partial}{\partial R} + \frac{\cos\theta}{R} \frac{\partial}{\partial \theta}. \tag{2}$$

In the moving coordinate system the material time derivative at a material point has the form

$$\dot{(\bullet)} = \left\{ \frac{\partial}{\partial t} - \dot{a}(t) \frac{\partial}{\partial x_1} \right\} (\bullet). \tag{3}$$

The equation of motion is

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i, j = 1, 2), \tag{4}$$

where ρ is the mass density of the material. Hereafter, the summation convention for repeated subscripts is used unless a different specification is given.

2.2. Constitutive relationship

As in Part 1, the constitutive relation of the elastic–plastic–viscoplastic material incorporated with damage can be described by

$$\begin{aligned} \dot{\varepsilon}_{ij} &= (\dot{\varepsilon}^{\mathrm{e}}_{ij})_{\mathrm{eff}} + (\dot{\varepsilon}^{\mathrm{p}}_{ij})_{\mathrm{eff}} + (\dot{\varepsilon}^{\mathrm{vp}}_{ij})_{\mathrm{eff}} \\ &= \frac{1+v}{E}(s_{ij})_{\mathrm{eff}} + \frac{1-2v}{3E}(\dot{\sigma}_{kk})_{\mathrm{eff}}\delta_{ij} + \frac{3}{2}B_0(\bar{\sigma})_{\mathrm{eff}}^{(1/N)-2}(s_{ij})_{\mathrm{eff}}(\dot{\bar{\sigma}})_{\mathrm{eff}} + \frac{3}{2}B(\bar{\sigma})_{\mathrm{eff}}^{n-1}(s_{ij})_{\mathrm{eff}}\left\{(1+q)B\int_{t_0}^{t}(\bar{\sigma})_{\mathrm{eff}}^{n}\,\mathrm{d}\tau\right\}^{-(q/(1+q))}, \end{aligned} \tag{5}$$

where

$$(\cdot)_{\mathrm{eff}} = \frac{(\cdot)}{1-D} \quad \text{and} \quad (\dot{\cdot})_{\mathrm{eff}} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{(\cdot)}{1-D}\right\} \tag{6}$$

in which D is an isotropic damage variable bounded by 0 and 1, with $D = 0$ representing undamaged state of the medium and $D = 1$ indicating the complete failure at a material meso-element. In Eq. (5), $(\dot{\varepsilon}^{\mathrm{e}}_{ij})_{\mathrm{eff}}$ represents the effective elastic strain rate, $(\dot{\varepsilon}^{\mathrm{p}}_{ij})_{\mathrm{eff}}$ the effective plastic strain rate, and $(\dot{\varepsilon}^{\mathrm{vp}}_{ij})_{\mathrm{eff}}$ the effective viscoplastic strain rate; E is Young's modulus, v Poisson's ratio, n the creep exponent, B a temperature-dependent material coefficient, N the hardening exponent of plasticity, B_0 a material constant related to the yield stress by $k\sigma_{\mathrm{y}}^{1-(1/N)}/E$, with k being a material constant; q is a constant for predicting primary creep $(q > 0)$, secondary creep $(q = 0)$, or tertiary creep $(q < 0)$; σ_{ij} denotes the stresses, s_{ij} $(= \sigma_{ij} - \sigma_{kk}\delta_{ij}/3)$ is the derviatoric stresses and $\bar{\sigma}$ the equivalent stress defined by

$$\bar{\sigma} = \left[\tfrac{3}{2}s_{ij}s_{ij}\right]^{1/2} \quad (i, j = 1, 2, 3). \tag{7}$$

2.3. Kinetic evolution equation of damage

The kinetic evolution equation of the quasi-brittle damage is taken as (Lemaitre, 1985, 1992)

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{2ES_\mathrm{D}}\left[\frac{2}{3}(1+v)\bar{\sigma}^2 + \frac{1}{3}(1-2v)\sigma_{kk}^2\right]\frac{\mathrm{d}P}{\mathrm{d}t}, \tag{8}$$

where S_D is a material constant, and $\dot{P} = \mathrm{d}P/\mathrm{d}t$ is the equivalent plastic strain rate defined by

$$\dot{P} = \frac{\mathrm{d}P}{\mathrm{d}t} \equiv \left[\frac{2}{3}\dot{\varepsilon}_{ij}^\mathrm{p}\dot{\varepsilon}_{ij}^\mathrm{p}\right]^{1/2} \tag{9}$$

As demonstrated by Eq. $(3)_\mathrm{c}$ of Part 1

$$\dot{\varepsilon}_{ij}^\mathrm{p} = B_0\bar{\sigma}^{(1/N)-2}\dot{\bar{\sigma}}s_{ij} \tag{10}$$

with Eq. (8) to Eq. (10) and mathematical manipulations, we have

$$D(1-D)^{\beta+2} = \frac{B_0}{2ES_\mathrm{D}}(c_1\bar{\sigma}^2 + c_2\sigma_{kk}^2)\bar{\sigma}^{\beta+1} \quad (k = 1,2), \tag{11}$$

where

$$\beta = \frac{1}{N} - 1, \tag{12}$$

$$c_1 = \frac{2(1+v)}{(\beta+4)}, \tag{13}$$

$$c_2 = \frac{(1-2v)}{(\beta+2)}. \tag{14}$$

The initial condition

$$D = 0 \quad \text{for} \quad \bar{\sigma} = 0 \tag{15}$$

is used when deriving Eq. (11).

In addition, since typically $N = 0.2$–0.3, β is a positive value usually larger than unity.

2.4. Initial and boundary conditions

The initial condition is that loads are suddenly applied to the cracked specimen at the time, t_0. According to the constitutive equation (5), the instantaneous response of the material is elastic. Besides, in the region where the yielding stress is reached, the instantaneous plastic response prevails. In the vicinity of the crack tip, it can be assumed that the full yield condition is met.

The boundary condition that we prescribed on the traction-free crack faces are $\sigma_{ij}n_i = 0$ $(i, j = 1, 2)$, where n_i is the normal vector on the crack face.

3. Field equations for asymptotic analysis

3.1. Dimensionless formulations

Let R^* be the characteristic dimension with which an inner zone (Region A $\oplus$ Region B in Fig. 1) and an outer region (Region C of Fig. 1) can be defined around the crack tip. The damage variable D in the outer region remains relatively small in comparison to unity. It should be pointed out that the definitions of "inner" and "outer" regions are relative. Even the outer region is still in the vicinity of the crack tip. In the following we shall seek the asymptotic solutions of Region C in Fig. 1, where small damage exists.

Let $\dot{a}^* = \dot{a}(t^*)$ be the characteristic crack velocity, where t^* is an arbitrary given time at which the crack velocity can be used as a scale. Then, the characteristic time for the problem is taken as $T^* = R^*/\dot{a}^*$.

Let σ^* be the corresponding characteristic stress. Consider that in the near-tip field the stress σ_{ij} has an asymptotic form

$$\sigma_{ij} \to R^{s_0}, \tag{16}$$

where s_0 is a constant to be determined. Hence, σ^* can be taken as

$$\sigma^* \propto R^{*s_0} \quad \text{or} \quad \sigma^* = G^* R^{*s_0}, \tag{17}$$

where G^* is a constant. The choice of G^* is not unique, but it must be characteristic of the angular distributions of stress within the scale of the characteristic time T^*. Let

$$\bar{x}_i = \frac{x_i}{R^*}, \quad \overline{R} = \frac{R}{R^*}, \quad \bar{t} = \frac{t}{T^*}, \quad \bar{\dot{u}}_i = \frac{\dot{u}_i}{\dot{a}^*}, \quad \bar{\dot{a}}(t) = \frac{\dot{a}(t)}{\dot{a}^*}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma^*}, \quad \bar{s}_{ij} = \frac{s_{ij}}{\sigma^*}, \quad \bar{\bar{\sigma}} = \frac{\bar{\sigma}}{\sigma^*}. \tag{18a}$$

Then,

$$\frac{\partial}{\partial t} = \frac{1}{T^*} \cdot \frac{\partial}{\partial \bar{t}} \quad \text{and} \quad \frac{\partial}{\partial x_j} = \frac{1}{R^*} \cdot \frac{\partial}{\partial \bar{x}_j}. \tag{18b}$$

All the non-dimensional physical quantities in Eq. (18a) have been scaled to the order of unity. Substituting Eqs. (18a) and (18b) into Eqs. (4) and (5), the equation of motion and the constitutive law can be, respectively, reduced to the dimensionless forms

$$\bar{\ddot{u}}_i = \alpha^{-1} \frac{\partial \bar{\sigma}_{ij}}{\partial \bar{x}_j}, \tag{19}$$

and

$$\frac{\partial \bar{\dot{u}}_i}{\partial \bar{x}_j} = \gamma_1^{\mathrm{e}} (\stackrel{\bullet}{\bar{s}}_{ij})_{\mathrm{eff}} + \gamma_2^{\mathrm{e}} (\stackrel{\bullet}{\bar{\sigma}}_{kk})_{\mathrm{eff}} \delta_{ij} + \kappa^{\mathrm{p}} (\bar{\bar{\sigma}})_{\mathrm{eff}}^{\beta+1} (\bar{s}_{ij})_{\mathrm{eff}} (\stackrel{\bullet}{\bar{\bar{\sigma}}})_{\mathrm{eff}} + \kappa^{\mathrm{vp}} (\bar{\bar{\sigma}})_{\mathrm{eff}}^{n-1} (\bar{s}_{ij})_{\mathrm{eff}} \left[\int_{t_0}^{t} (\bar{\bar{\sigma}})_{\mathrm{eff}}^{n} \, \mathrm{d}\tau \right]^{-(q/(1+q))}, \quad (i,j,k = 1,2) \tag{20}$$

where the dimensionless coefficients, α, γ_1^{e}, γ_2^{e}, κ^{p} and κ^{vp}, are expressed by

$$\alpha = \frac{\rho \dot{a}^{*2}}{G^* R^{*s_0}}, \tag{21}$$

$$\gamma_1^{\mathrm{e}} = \frac{1+v}{E} G^* R^{*s_0}, \tag{22}$$

$$\gamma_2^{\mathrm{e}} = \frac{1-2v}{3E} G^* R^{*s_0}, \tag{23}$$

$$\kappa^{\mathrm{p}} = \frac{3}{2} B_0 G^* [R^{*s_0}]^{1+\beta}, \tag{24}$$

$$\kappa^{\mathrm{vp}} = \frac{3}{2} B[B(1+q)]^{-q/(1+q)} [G^* R^{*s_0}]^{n/(1+q)} \left(\frac{R^*}{\dot{a}^*}\right)^{1/(1+q)}. \tag{25}$$

Correspondingly, the damage evolution equation (11) is normalised to

$$D(1-D)^{\beta+2} = \delta(c_1\bar{\bar{\sigma}}^2 + c_2\bar{\sigma}_{kk}^2)\bar{\bar{\sigma}}^{1+\beta} \quad (k=1,2), \tag{26}$$

where

$$\delta = \frac{B_0}{2ES_{\mathrm{D}}} (G^* R^{*s_0})^{\beta+3}. \tag{27}$$

In the case of small damage, Eq. (26) can be reduced to

$$D \doteq \delta\left(c_1\bar{\bar{\sigma}}^2 + c_2\bar{\sigma}_{kk}^2\right)\bar{\bar{\sigma}}^{1+\beta}. \tag{28}$$

The physical quantities on the right hand side of Eq. (28) have been scaled to the order of unity except the dimensionless coefficient, δ, which is of the order of the characteristic damage. Therefore,

$$D \ll 1 \Longleftrightarrow \delta \ll 1. \tag{29}$$

From Eq. (27) it can be identified that the condition $\delta \ll 1$ requires

$$R^* \gg \left[G^*\left(\frac{B_0}{2ES_{\mathrm{D}}}\right)^{1/(3+\beta)}\right]^{1/(|s_0|)}, \tag{30}$$

which, in turn, defines the small damage condition.

3.2. Perturbation expansion

Since in the following discussion only the non-dimensional quantities are involved, for convenience, we remove the bars from the barred parameters. However, they are still dimensionless.

Consider the dimensionless constant δ as a small quantity and follow the regular perturbation procedure, then

$$\dot{u}_i = \dot{u}_i^{(0)} + \delta\dot{u}_i^{(1)} + \delta^2\dot{u}_i^{(2)} + \cdots = \sum_{m=0}^{\infty} \delta^m \dot{u}_i^{(m)}, \tag{31}$$

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \delta\sigma_{ij}^{(1)} + \delta^2\sigma_{ij}^{(2)} + \cdots = \sum_{m=0}^{\infty} \delta^m \sigma_{ij}^{(m)}, \tag{32}$$

$$D = D^{(0)} + \delta D^{(1)} + \delta^2 D^{(2)} + \cdots = \sum_{m=0}^{\infty} \delta^m D^{(m)} \quad (i,j=1,2). \tag{33}$$

Substituting Eqs. (31)–(33) in Eqs. (19) and (20), we can obtain a system of asymptotic equations. Indeed, by expanding all the physical quantities in Eqs. (19) and (20) in power series with respect to δ, and equating those terms with the same order of δ, a series of asymptotic equations can be derived. For instance, the δ^0 order asymptotic equations are

$$\ddot{u}_i^{(0)} = \alpha^{-1}\frac{\partial \sigma_{ij}^{(0)}}{\partial x_i}, \tag{34}$$

$$\frac{\partial \dot{u}_i^{(0)}}{\partial x_j} = \gamma_1^{e}\dot{s}_{ij}^{(0)} + \gamma_2^{e}\dot{\sigma}_{kk}^{(0)}\delta_{ij} + \kappa^{p}(\bar{\sigma}^{(0)})^{\beta+1}s_{ij}^{(0)}\dot{\bar{\sigma}}^{(0)} + \kappa^{vp}(\bar{\sigma}^{(0)})^{n-1}s_{ij}^{(0)}\left[\int_{t_0}^{t}(\bar{\sigma}^{(0)})^{n}\,d\tau\right]^{-(q/(1+q))} \quad (i,j,k=1,2), \tag{35}$$

where

$$s_{ij}^{(0)} = \sigma_{ij}^{(0)} - \tfrac{1}{3}\sigma_{kk}^{(0)}\delta_{ij}, \tag{36}$$

$$\bar{\sigma}^{(0)} = \left[(\sigma_{11}^{(0)})^2 + (\sigma_{22}^{(0)})^2 - \sigma_{11}^{(0)}\sigma_{22}^{(0)} + 3(\sigma_{12}^{(0)})^2\right]^{1/2}. \tag{37}$$

In deriving the above equations, the condition, $D^{(0)} = 0$, has been used, which corresponds to the undamaged state of the virgin material.

Eqs. (34) and (35), together with the initial and boundary conditions addressed previously, describe the problem of a crack propagating in an elastic–power-law–plastic–viscoplastic solid without damage.

The δ^1 order asymptotic equations are

$$\ddot{u}_i^{(1)} = \alpha^{-1}\frac{\partial \sigma_{ij}^{(1)}}{\partial x_j}, \tag{38}$$

$$\begin{aligned}\frac{\partial v_i^{(1)}}{\partial x_j} = {}& \gamma_1^{e}\frac{d}{dt}\left[s_{ij}^{(1)} + s_{ij}^{(0)}D^{(1)}\right] + \gamma_2^{e}\frac{d}{dt}\left[\sigma_{kk}^{(1)} + \sigma_{kk}^{(0)}D^{(1)}\right]\delta_{ij} + \kappa^{p}\left(\bar{\sigma}^{(0)}\right)^{\beta}\Bigg\{(\beta+1)s_{ij}^{(0)}\left[\bar{\sigma}^{(1)} + \bar{\sigma}^{(0)}D^{(1)}\right]\frac{d\bar{\sigma}^{(0)}}{dt} \\ & + \bar{\sigma}^{(0)}\left[s_{ij}^{(1)} + s_{ij}^{(0)}D^{(1)}\right]\frac{d\bar{\sigma}^{(0)}}{dt} + \bar{\sigma}^{(0)}s_{ij}^{(0)}\frac{d}{dt}\left[\bar{\sigma}^{(1)} + \bar{\sigma}^{(0)}D^{(1)}\right]\Bigg\} + \kappa^{vp}\left(\bar{\sigma}^{(0)}\right)^{n-2} \\ & \times\Bigg\{\left[\int_{t_0}^{t}\left(\bar{\sigma}^{(0)}\right)^{n}d\tau\right]^{-(q/(1+q))}\left[(n-1)s_{ij}^{(0)}\left(\bar{\sigma}^{(1)} + \bar{\sigma}^{(0)}D^{(1)}\right) + \bar{\sigma}^{(0)}\left(s_{ij}^{(1)} + s_{ij}^{(0)}D^{(1)}\right)\right] - \frac{qn}{1+q}\left(\bar{\sigma}^{(0)}\right)^{n-1}s_{ij}^{(0)} \\ & \times\left[\int_{t_0}^{t}(\bar{\sigma}^{(0)})^{n}\,d\tau\right]^{-(2q+1)/(1+q)}\left[\int_{t_0}^{\tilde{t}}\left(\bar{\sigma}^{(0)}\right)^{n-1}\left(\bar{\sigma}^{(1)} + \bar{\sigma}^{(0)}D^{(1)}\right)d\tau\right]\Bigg\} \quad (i,j,k=1,2),\end{aligned} \tag{39}$$

where

$$s_{ij}^{(1)} = \sigma_{ij}^{(1)} - \tfrac{1}{3}\sigma_{kk}^{(1)}\delta_{ij}, \tag{40}$$

$$\bar{\sigma}^{(1)} = \frac{1}{2\bar{\sigma}^{(0)}}\left[\left(2\sigma_{11}^{(0)} - \sigma_{22}^{(0)}\right)\sigma_{11}^{(1)} + \left(2\sigma_{22}^{(0)} - \sigma_{11}^{(0)}\right)\sigma_{22}^{(1)} + 6\sigma_{12}^{(0)}\sigma_{12}^{(1)}\right], \tag{41}$$

$$D^{(1)} = \left\lfloor c_1\left(\bar{\sigma}^{(0)}\right)^2 + c_2\left(\sigma_{kk}^{(0)}\right)^2\right\rfloor\left(\bar{\sigma}^{(0)}\right)^{\beta+1}. \tag{42}$$

The third-order and even higher-order asymptotic equations can be obtained by a similar procedure.

3.3. Field equations for the regime with small damage

3.3.1. First-order asymptotic solution

For the first-order asymptotic equations (34) and (35), according to fracture mechanics, the solution takes the form below. That is,

$$\dot{u}_i^{(0)}(R,\theta,t) = R^{s_0}\dot{U}_i^{(0)}(\theta,t) \tag{43}$$

and

$$\sigma_{ij}^{(0)}(R,\theta,t) = R^{s_0}\Sigma_{ij}^{(0)}(\theta,t) \quad (i,j=1,2). \tag{44}$$

Here, s_0, $\dot{U}_i^{(0)}(\theta,t)$, and $\Sigma_{ij}^{(0)}(\theta,t)$ have to be determined.

Note that in terms of the differential operator defined by Eq. (3), we can see that $\partial/\partial t$ is one order lower than $-\dot{a}(t)\partial/\partial x_1$. Thus,

$$(\dot{\,}) \doteq -\dot{a}(t)\frac{\partial}{\partial x_1}(\cdot). \tag{45}$$

Using the coordinate transform Eqs. (1) and (2) after substitution of Eqs. (43) and (44) into Eq. (34), the equation of motion is reduced to

$$-\dot{a}(t)L_1^{(s_0)}\dot{U}_i^{(0)}(\theta,t) = \alpha^{-1}L_j^{(s_0)}\Sigma_{ij}^{(0)}(\theta,t), \tag{46}$$

where $L_j^{(s_0)}$ $(j=1,2)$ is the differential operator defined by

$$L_1^{(s_0)} = s_0\cos\theta - \sin\theta\frac{\partial}{\partial\theta} \quad \text{and} \quad L_2^{(s_0)} = s_0\sin\theta + \cos\theta\frac{\partial}{\partial\theta} \tag{47}$$

for $j=1$ and 2, respectively.

Expand $\dot{\varepsilon}_{ij}$ as

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial\dot{u}_i}{\partial x_j} + \frac{\partial\dot{u}_j}{\partial x_i}\right) = \frac{1}{2}\left[L_i^{(s_0)}\dot{U}_j^{(0)}(\theta,t) + L_j^{(s_0)}\dot{U}_i^{(0)}(\theta,t)\right]R^{s_0-1}. \tag{48}$$

Then, elimination of $\dot{\varepsilon}_{ij}$ in Eq. (35) with Eq. (48) after substitution of Eqs. (43)–(45) into Eq. (35) produces,

$$\begin{aligned}
&\left[L_i^{(s_0)}\dot{U}_j^{(0)}(\theta,t) + L_j^{(s_0)}\dot{U}_i^{(0)}(\theta,t) + \gamma_1^{\mathrm{e}}\dot{a}(t)L_1^{(s_0)}S_{ij}^{(0)}(\theta,t) + \gamma_2^{\mathrm{e}}\dot{a}(t)L_1^{(s_0)}\Sigma_{kk}^{(0)}(\theta,t)\delta_{ij}\right]R^{s_0-1} \\
&\quad = -\dot{a}(t)\kappa^{\mathrm{p}}\left[\overline{\Sigma}^{(0)}(\theta,t)\right]^{\beta+1}S_{ij}^{(0)}(\theta,t)L_1^{(s_0)}\overline{\Sigma}^{(0)}(\theta,t)R^{(\beta+2)s_0-1} \\
&\qquad + \kappa^{\mathrm{vp}}\left[\overline{\Sigma}^{(0)}(\theta,t)\right]^{n-1}S_{ij}^{(0)}(\theta,t)\left\{\int_{t_0}^{t}\left[\overline{\Sigma}^{(0)}(\theta,t)\right]^{n}\mathrm{d}\tau\right\}^{-(q/(1+q))}R^{ns_0/(1+q)},
\end{aligned} \tag{49}$$

where

$$S_{ij}^{(0)}(\theta,t) = \Sigma_{ij}^{(0)}(\theta,t) - \tfrac{1}{3}\Sigma_{kk}^{(0)}(\theta,t)\delta_{ij}, \tag{50}$$

$$\overline{\Sigma}^{(0)}(\theta,t) = \left[\left(\Sigma_{11}^{(0)}(\theta,t)\right)^2 + \left(\Sigma_{22}^{(0)}(\theta,t)\right)^2 - \Sigma_{11}^{(0)}(\theta,t)\cdot\Sigma_{22}^{(0)}(\theta,t) + 3\left(\Sigma_{12}^{(0)}(\theta,t)\right)^2\right]^{1/2}. \tag{51}$$

Following the method used in Part 1, with Eq. (49) we can determine which term(s) dominates the crack tip behaviour in Region C (see Fig. 1). Suppose the term associated with viscoplasticity in Eq. (49) can be ignored in comparison to that relevant to plasticity. Obviously, this case describes the problem of a crack propagating in an elastic–power-law–plastic solid. Equating the exponents of R on both sides of Eq. (49) requires that

$$s_0 \equiv 0. \tag{52}$$

Eq. (52) leads to a trivial solution. Thus, it can be concluded that such form of the asymptotic solution expressed by Eqs. (43) and (44) does not exist for the problem of a crack propagating in a purely elastic–power-law–plastic material. Note that a similar conclusion was obtained for mode III dynamic crack by means of a fully numerical verification (Gao et al., 1983). We obtained a similar result in Part 1, but what we discussed there was a damage-related elastic–plastic solid. This conclusion obtained here is, as addressed in Part 1, also restricted to such case in which the deformation theory is available. A further discussion relevant to this issue is carried out in another work (Lu et al., 2001b), in which the elastic unloading and (possible) plastic re-unloading processes are considered.

Assume now that both plasticity and viscoplasticity terms are equally important. Following the same procedure used above, we may see that no mathematically consistent result can be obtained. Thus, in the concerned regime (Region C) the stress and strain rate fields cannot be the mixed plastic–viscoplastic type.

To seek non-trivial solutions let us turn to the last case in which the plasticity term is removed but the term of viscoplasticity is retained on the right hand side of Eq. (49). Then, Eq. (49) becomes

$$\begin{aligned}&\left[L_i^{(s_0)}\dot{U}_j^{(0)}(\theta,t)+L_j^{(s_0)}\dot{U}_i^{(0)}(\theta,t)+\gamma_1^{\mathrm{e}}\dot{a}(t)L_1^{(s_0)}S_{ij}^{(0)}(\theta,t)+\gamma_2^{\mathrm{e}}\dot{a}(t)L_1^{(s_0)}\Sigma_{kk}^{(0)}(\theta,t)\delta_{ij}\right]R^{s_0-1}\\&\quad=\kappa^{\mathrm{vp}}\left[\overline{\Sigma}^{(0)}(\theta,t)\right]^{n-1}S_{ij}^{(0)}(\theta,t)\left\{\int_{t_0}^{t}\left[\overline{\Sigma}^{(0)}(\theta,t)\right]^{n}\mathrm{d}\tau\right\}^{-(q/(1+q))}R^{ns_0/(1+q)}.\end{aligned} \tag{53}$$

By equating the exponents of R on both sides of Eq. (53), we have

$$s_0=-\left(\frac{n}{1+q}-1\right)^{-1}. \tag{54}$$

Eqs. (46) and (53) compose of a set of integral–differential equations. Now we seek the form of solutions:

$$\dot{U}_i^{(0)}(\theta,t)=A^{(0)}(t)\widetilde{\dot{U}}_i^{(0)}(\theta)\quad\text{and}\quad\Sigma_{ij}^{(0)}(\theta,t)=G^{(0)}(t)\widetilde{\Sigma}_{ij}^{(0)}(\theta), \tag{55}$$

where $A^{(0)}(t)$, $G^{(0)}(t)$, $\widetilde{\dot{U}}_i^{(0)}(\theta)$, and $\widetilde{\Sigma}_{ij}^{(0)}(\theta)$ are to be determined.

Substitution of Eq. (55) into Eq. (46) gives

$$\frac{\alpha\dot{a}(t)A^{(0)}(t)}{G^{(0)}(t)}=-\frac{L_j^{(s_0)}\widetilde{\Sigma}_{1j}^{(0)}(\theta)}{L_1^{(s_0)}\widetilde{\dot{U}}_1^{(0)}(\theta)}, \tag{56}$$

$$\frac{\alpha\dot{a}(t)A^{(0)}(t)}{G^{(0)}(t)}=-\frac{L_j^{(s_0)}\widetilde{\Sigma}_{2j}^{(0)}(\theta)}{L_1^{(s_0)}\widetilde{\dot{U}}_2^{(0)}(\theta)}. \tag{57}$$

Note that the left-hand sides of Eqs. (56) and (57) are functions of time t while the right-hand sides are functions of θ. Therefore, they must be identical to a constant, $k_1^{(0)}$. Hence,

$$\alpha\dot{a}(t)A^{(0)}(t)=k_1^{(0)}G^{(0)}(t), \tag{58}$$

and

$$k_1^{(0)}L_1^{(s_0)}\widetilde{\dot{U}}_1^{(0)}(\theta)+L_j^{(s_0)}\widetilde{\Sigma}_{1j}^{(0)}(\theta)=0, \tag{59}$$

$$k_1^{(0)} L_1^{(s_0)} \widetilde{U}_2^{(0)}(\theta) + L_j^{(s_0)} \widetilde{\Sigma}_{2j}^{(0)}(\theta) = 0. \tag{60}$$

$k_1^{(0)}$ has to be determined from the global solution.

Substitution of Eqs. (55) and (58) in Eq. (53) yields

$$\frac{A^{(0)}(t)}{\kappa^{\mathrm{vp}}[G^{(0)}(t)]^n \left\{ \int_{t_0}^t [G^{(0)}(t)]^n \, \mathrm{d}\tau \right\}^{-(q/(1+q))}} = \frac{\left[\widetilde{\overline{\Sigma}}^{(0)}(\theta)\right]^{(n/(1+q))-1} \widetilde{S}_{11}^{(0)}(\theta)}{L_1^{(s_0)} \widetilde{U}_1^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{T}}^2 L_1^{(s_0)} \widetilde{S}_{11}^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \xi \beta_{\mathrm{v}}^2 L_1^{(s_0)} \widetilde{\Sigma}_{kk}^{(0)}(\theta)}, \tag{61}$$

$$\frac{A^{(0)}(t)}{\kappa^{\mathrm{vp}}[G^{(0)}(t)]^n \left\{ \int_{t_0}^t [G^{(0)}(t)]^n \, \mathrm{d}\tau \right\}^{-(q/(1+q))}} = \frac{\left[\widetilde{\overline{\Sigma}}^{(0)}(\theta)\right]^{(n/(1+q))-1} \widetilde{S}_{22}^{(0)}(\theta)}{L_1^{(s_0)} \widetilde{U}_1^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{T}}^2 L_1^{(s_0)} \widetilde{S}_{22}^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \xi \beta_{\mathrm{v}}^2 L_1^{(s_0)} \widetilde{\Sigma}_{kk}^{(0)}(\theta)}, \tag{62}$$

$$\frac{A^{(0)}(t)}{\kappa^{\mathrm{vp}}[G^{(0)}(t)]^n \left\{ \int_{t_0}^t [G^{(0)}(t)]^n \, \mathrm{d}\tau \right\}^{-(q/(1+q))}} = \frac{\left[\widetilde{\overline{\Sigma}}^{(0)}(\theta)\right]^{(n/(1+q))-1} \widetilde{S}_{12}^{(0)}(\theta)}{L_1^{(s_0)} \widetilde{U}_1^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \xi \beta_{\mathrm{T}}^2 L_1^{(s_0)} \widetilde{S}_{12}^{(0)}(\theta)}, \tag{63}$$

where

$$\beta_{\mathrm{T}}^2 = \left(\frac{\dot{a}(t)}{\dot{a}^*}\right)\left(\frac{\dot{a}^*}{c_{\mathrm{T}}}\right)^2 \quad \text{and} \quad \beta_{\mathrm{v}}^2 = \left(\frac{\dot{a}(t)}{\dot{a}^*}\right)\left(\frac{\dot{a}^*}{c_{\mathrm{v}}}\right)^2 \tag{64}$$

in which

$$c_{\mathrm{T}} = \sqrt{\frac{E}{2(1+v)\rho}} \quad \text{and} \quad c_{\mathrm{V}} = \sqrt{\frac{(1-v)}{(1+v)} \frac{E}{\rho(1-2v)}} \tag{65}$$

are the velocities of the shear and dilatation waves, respectively. Here, $\xi = (1-v)/(1+v)$.

For steady crack propagation, $\dot{a}(t) = \dot{a} =$ constant. Then, the left and right hand side of Eqs. (61)–(63) are functions of t and θ, respectively. Thus, they must be identical to a constant. Therefore, we obtain

$$A^{(0)}(t) = k_2^{(0)} \kappa^{\mathrm{vp}} \left[G^{(0)}(t)\right]^n \left\{ \int_{t_0}^t \left[G^{(0)}(t)\right]^n \mathrm{d}\tau \right\}^{-(q/(1+q))}, \tag{66}$$

$$L_1^{(s_0)} \widetilde{U}_1^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{T}}^2 L_1^{(s_0)} \widetilde{S}_{11}^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{v}}^2 L_1^{(s_0)} \widetilde{\Sigma}_{kk}^{(0)}(\theta) = \left(k_2^{(0)}\right)^{-1} \left[\widetilde{\overline{\Sigma}}^{(0)}(\theta)\right]^{(n/(1+q))-1} \widetilde{S}_{11}^{(0)}(\theta), \tag{67}$$

$$L_2^{(s_0)} \widetilde{U}_2^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{T}}^2 L_1^{(s_0)} \widetilde{S}_{22}^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{v}}^2 L_1^{(s_0)} \widetilde{\Sigma}_{kk}^{(0)}(\theta) = \left(k_2^{(0)}\right)^{-1} \left[\widetilde{\overline{\Sigma}}^{(0)}(\theta)\right]^{(n/(1+q))-1} \widetilde{S}_{22}^{(0)}(\theta), \tag{68}$$

$$L_1^{(s_0)} \widetilde{U}_2^{(0)}(\theta) + L_2^{(s_0)} \widetilde{U}_1^{(0)}(\theta) + \left(k_1^{(0)}\right)^{-1} \beta_{\mathrm{T}}^2 L_1^{(s_0)} \widetilde{S}_{12}^{(0)}(\theta) = 2\left(k_2^{(0)}\right)^{-1} \left[\widetilde{\overline{\Sigma}}^{(0)}(\theta)\right]^{(n/(1+q))-1} \widetilde{S}_{12}^{(0)}(\theta), \tag{69}$$

where, the constant $k_2^{(0)}$, like $k_1^{(0)}$, is dependent on the global solution and can not be determined by local analysis alone.

Solution to Eqs. (58) and (66) gives

$$A^{(0)}(t)=\begin{cases}\frac{k_1^{(0)}}{\alpha\dot{a}}\left[\left(G^{(0)}(t_0)\right)^{1-\eta}+\frac{1-\eta}{\varsigma}(t-t_0)\right]^{1/(1-\eta)}, & (q\neq 0)\\ k_2^{(0)}\left(\frac{k_1^{(0)}}{k_2^{(0)}\alpha\dot{a}\kappa^{\mathrm{vp}}}\right)^{n/(n-1)}, & (q=0)\end{cases} \tag{70}$$

and

$$G^{(0)}(t)=\begin{cases}\left[\left(G^{(0)}(0)\right)^{1-\eta}+\frac{1-\eta}{\varsigma}(t-t_0)\right]^{1/(1-\eta)}, & (q\neq 0)\\ \left(\frac{k_1^{(0)}}{k_2^{(0)}\alpha\dot{a}\kappa^{\mathrm{vp}}}\right)^{1/(n-1)}, & (q=0)\end{cases} \tag{71}$$

where

$$\eta=n-\frac{(n-1)(1+q)}{q}+1 \quad \text{and} \quad \varsigma=\frac{(n-1)(1+q)}{q}\left\{\left(\frac{k_2^{(0)}}{k_1^{(0)}}\right)\alpha\dot{a}\kappa^{\mathrm{vp}}\right\}^{(1+q)/q}. \tag{72}$$

It can be seen from Eqs. (56), (57) and (67) to Eq. (69) that in the elastic–viscoplastic solid (the plastic effect has been discarded according to the above asymptotic analysis), the elastic effect has the same order as the inertia effect, and therefore it cannot be neglected in general. However, it was shown in Part 1 that the effective viscoplasticity controls the crack tip field. Hence, it is interesting to exclude the elasticity terms here, and to see how the two viscoplastic stress fields vary in the regions of large and small damage, respectively. For this purpose, we confine our study to the special case that $(k_1^{(0)})^{-1}\beta_{\mathrm{T}}^2$ and $(k_1^{(0)})^{-1}\beta_{\mathrm{v}}^2$ in Eqs. (66)–(69) are much smaller than unity so that the corresponding terms can be neglected in comparison to the viscoplastic terms. Hence, Eqs. (59), (60), and (67)–(69) become

$$k_1^{(0)}\sin\theta\left[\tilde{\dot{U}}_1^{(0)}(\theta)\right]'+\sin\theta\left[\tilde{\Sigma}_{11}^{(0)}(\theta)\right]'-\cos\theta\left[\tilde{\Sigma}_{12}^{(0)}(\theta)\right]'=\varphi_1^{(0)}(\theta), \tag{73}$$

$$k_1^{(0)}\sin\theta\left[\tilde{\dot{U}}_2^{(0)}(\theta)\right]'-\cos\theta\left[\tilde{\Sigma}_{22}^{(0)}(\theta)\right]'+\sin\theta\left[\tilde{\Sigma}_{12}^{(0)}(\theta)\right]'=\varphi_2^{(0)}(\theta), \tag{74}$$

$$k_2^{(0)}\sin\theta\left[\tilde{\dot{U}}_1^{(0)}(\theta)\right]'=\varphi_3^{(0)}(\theta), \tag{75}$$

$$k_2^{(0)}\cos\theta\left[\tilde{\dot{U}}_2^{(0)}(\theta)\right]'=\varphi_4^{(0)}(\theta), \tag{76}$$

$$k_2^{(0)}s_1\sin\theta\left[\tilde{\dot{U}}_1^{(0)}(\theta)\right]'-k_2^{(0)}s_1\cos\theta\left[\tilde{\dot{U}}_1^{(0)}(\theta)\right]'=\varphi_5^{(0)}(\theta), \tag{77}$$

where

$$\varphi_1^{(0)}(\theta)=k_1^{(0)}s_0\cos\theta\,\tilde{\dot{U}}_1^{(0)}(\theta)+s_0\sin\theta\,\tilde{\Sigma}_{11}^{(0)}(\theta)+s_0\sin\theta\,\tilde{\Sigma}_{12}^{(0)}(\theta), \tag{78}$$

$$\varphi_2^{(0)}(\theta)=k_1^{(0)}s_0\cos\theta\,\tilde{\dot{U}}_2^{(0)}(\theta)+s_0\sin\theta\,\tilde{\Sigma}_{12}^{(0)}(\theta)+s_0\sin\theta\,\tilde{\Sigma}_{22}^{(0)}(\theta), \tag{79}$$

$$\varphi_3^{(0)}(\theta)=k_2^{(0)}s_0\cos\tilde{\dot{U}}_1^{(0)}(\theta)-\left(\tilde{\overline{\Sigma}}^{(0)}(\theta)\right)^{\lambda-1}\tilde{S}_{11}^{(0)}(\theta), \tag{80}$$

$$\varphi_4^{(0)}(\theta)=k_2^{(0)}s_0\sin\theta\,\tilde{\dot{U}}_2^{(0)}(\theta)-\left(\tilde{\overline{\Sigma}}^{(0)}(\theta)\right)^{\lambda-1}\tilde{S}_{22}^{(0)}(\theta), \tag{81}$$

$$\varphi_5^{(0)}(\theta) = -k_2^{(0)} s_0 \sin\theta\, \tilde{\dot{U}}_1^{(0)}(\theta) - k_2^{(0)} s_0 \cos\theta\, \tilde{\dot{U}}_2^{(0)}(\theta) + 2\left(\tilde{\overline{\Sigma}}^{(0)}(\theta)\right)^{\lambda-1} \tilde{S}_{12}^{(0)}(\theta) \tag{82}$$

in which $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}\theta$ and $\lambda = n/(1+q)$.

The corresponding boundary conditions at $\theta = 0$ and $\theta = \pi$ are given below. By virtue of symmetry, the following conditions hold at $\theta = 0$

$$\tilde{\dot{U}}_2^{(0)}(0) = 0 \quad \text{and} \quad \tilde{\Sigma}_{12}^{(0)}(0) = 0. \tag{83}$$

At $\theta = \pi$ the free surface conditions provide

$$\tilde{\Sigma}_{12}^{(0)}(\pi) = 0 \quad \text{and} \quad \tilde{\Sigma}_{22}^{(0)}(\pi) = 0. \tag{84}$$

There are five unknowns, $\tilde{\dot{U}}_1^{(0)}$, $\tilde{\dot{U}}_2^{(0)}$, $\tilde{\Sigma}_{11}^{(0)}$, $\tilde{\Sigma}_{12}^{(0)}$ $(= \tilde{\Sigma}_{21}^{(0)})$ and $\tilde{\Sigma}_{22}^{(0)}$, in Eqs. (73)–(77), yet Eqs. (83) and (84) only supply four boundary conditions. Another boundary condition can be supplemented with the regularity condition of Eq. (75) which requires that

$$\varphi_3^{(0)}(0) = 0 \tag{85}$$

at $\theta = 0$.

It can be readily verified that the symmetric requirement of the velocity at $\theta = 0$,

$$\tilde{\dot{U}}_1'^{(0)}(0) = 0 \tag{86}$$

can be automatically satisfied.

3.3.2. Second-order asymptotic solution

For the second-order analysis, in the case that the elastic effect is ignored, we let

$$\dot{u}_i^{(1)}(\theta, t) = A^{(1)}(t) R^{s_1} \tilde{\dot{U}}_i^{(1)}(\theta) \tag{87}$$

and

$$\tilde{\sigma}_{ij}^{(1)}(\theta, t) = G^{(1)}(t) R^{s_1} \tilde{\Sigma}_{ij}^{(1)}(\theta). \tag{88}$$

Then, substituting Eqs. (87) and (88) in Eqs. (38) and (39) and following the same procedure used in derivation of first order approximation, we have

$$k_1^{(1)} \sin\theta \left[\tilde{\dot{U}}_1^{(1)}(\theta)\right]' + \sin\theta \left[\tilde{\Sigma}_{11}^{(1)}(\theta)\right]' - \cos\theta \left[\tilde{\Sigma}_{12}^{(1)}(\theta)\right]' = \varphi_1^{(1)}(\theta), \tag{89}$$

$$k_1^{(1)} \sin\theta \left[\tilde{\dot{U}}_2^{(1)}(\theta)\right]' - \cos\theta \left[\tilde{\Sigma}_{22}^{(1)}(\theta)\right]' + \sin\theta \left[\tilde{\Sigma}_{12}^{(1)}(\theta)\right]' = \varphi_2^{(1)}(\theta), \tag{90}$$

$$k_2^{(1)} \sin\theta \left[\tilde{\dot{U}}_1^{(1)}(\theta)\right]' = \varphi_3^{(1)}(\theta), \tag{91}$$

$$k_2^{(1)} \cos\theta \left[\tilde{\dot{U}}_2^{(1)}(\theta)\right]' = \varphi_4^{(1)}(\theta), \tag{92}$$

$$k_2^{(1)} s_1 \cos\theta \left[\tilde{\dot{U}}_1^{(1)}(\theta)\right]' + k_2^{(1)} s_1 \cos\theta \left[\tilde{\dot{U}}_2^{(1)}(\theta)\right]' + m_1 \left[\tilde{\Sigma}_{11}^{(1)}(\theta)\right]' + m_2 \left[\tilde{\Sigma}_{22}^{(1)}(\theta)\right]' + m_3 \left[\tilde{\Sigma}_{12}^{(1)}(\theta)\right]' = \varphi_5^{(0)}(\theta), \tag{93}$$

where

$$\varphi_1^{(1)} = k_1^{(1)} s_1 \cos\theta \, \tilde{\dot{U}}_1^{(1)}(\theta) + s_1 \cos\theta \, \widetilde{\Sigma}_{11}^{(1)}(\theta) + s_1 \sin\theta \, \widetilde{\Sigma}_{12}^{(1)}(\theta), \tag{94}$$

$$\varphi_2^{(1)} = k_1^{(1)} s_1 \cos\theta \, \tilde{\dot{U}}_2^{(1)}(\theta) + s_1 \cos\theta \, \widetilde{\Sigma}_{12}^{(1)}(\theta) + s_1 \sin\theta \, \widetilde{\Sigma}_{22}^{(1)}(\theta), \tag{95}$$

$$\varphi_3^{(1)} = k_2^{(1)} s_1 \cos\theta \, \tilde{\dot{U}}_1^{(1)}(\theta) - \Omega_{11}^{(1)}(\theta), \tag{96}$$

$$\varphi_4^{(1)} = -k_2^{(1)} s_1 \sin\theta \, \tilde{\dot{U}}_2^{(1)}(\theta) + \Omega_{22}^{(1)}(\theta), \tag{97}$$

$$\varphi_5^{(1)} = \frac{\partial}{\partial\theta}\left\{ \cos^2\theta\left[c_{11}^{(1)} + c_{11}^{(2)} + c_{11}^{(3)} + h_1\right] + \sin^2\theta\left[c_{22}^{(1)} + c_{22}^{(2)} + c_{22}^{(3)} + h_2\right] + \frac{\partial}{\partial\theta}\left[c_{12}^{(1)} + c_{12}^{(2)} + c_{12}^{(3)} + h_3\right]\right\} \tag{98}$$

in which

$$\Omega_{11}^{(1)} = \left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-2}\left[(\lambda-1)\widetilde{S}_{11}^{(0)}\widetilde{\overline{\Sigma}}^{(1)} + \widetilde{\overline{\Sigma}}^{(0)}\widetilde{S}_{11}^{(1)} + \lambda\widetilde{\overline{\Sigma}}^{(0)}\widetilde{S}_{11}^{(0)}D^{(1)}\right],$$

$$\Omega_{22}^{(1)} = \left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-2}\left[(\lambda-1)\widetilde{S}_{22}^{(0)}\widetilde{\overline{\Sigma}}^{(1)} + \widetilde{\overline{\Sigma}}^{(0)}\widetilde{S}_{22}^{(1)} + \lambda\widetilde{\overline{\Sigma}}^{(0)}\widetilde{S}_{22}^{(0)}D^{(1)}\right],$$

$$\Omega_{22}^{(1)} = \tfrac{1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-2}\left[(\lambda-1)\widetilde{S}_{12}^{(0)}\widetilde{\overline{\Sigma}}^{(1)} + \widetilde{\overline{\Sigma}}^{(0)}\widetilde{S}_{12}^{(1)} + \lambda\widetilde{\overline{\Sigma}}^{(0)}\widetilde{S}_{12}^{(0)}D^{(1)}\right],$$

$$c_{11}^{(1)} = \frac{\lambda-1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{11}^{(0)}\left[2\widetilde{\Sigma}_{11}^{(0)} - \widetilde{\Sigma}_{22}^{(0)}\right] + \frac{2}{3}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-1},$$

$$c_{22}^{(1)} = \frac{\lambda-1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{11}^{(0)}\left[2\widetilde{\Sigma}_{22}^{(0)} - \widetilde{\Sigma}_{11}^{(0)}\right] - \frac{1}{3}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-1},$$

$$c_{12}^{(1)} = 3(\lambda-1)\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{11}^{(0)}\widetilde{\Sigma}_{12}^{(0)},$$

$$c_{11}^{(2)} = \frac{\lambda-1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{22}^{(0)}\left[2\widetilde{\Sigma}_{11}^{(0)} - \widetilde{\Sigma}_{22}^{(0)}\right] - \frac{1}{3}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-1},$$

$$c_{22}^{(2)} = \frac{\lambda-1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{22}^{(0)}\left[2\widetilde{\Sigma}_{22}^{(0)} - \widetilde{\Sigma}_{11}^{(0)}\right] + \frac{2}{3}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-1},$$

$$c_{12}^{(1)} = 3(\lambda-1)\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{22}^{(0)}\widetilde{\Sigma}_{12}^{(0)},$$

$$c_{11}^{(3)} = \frac{\lambda-1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{12}^{(0)}\left[2\widetilde{\Sigma}_{11}^{(0)} - \widetilde{\Sigma}_{22}^{(0)}\right],$$

$$c_{22}^{(3)} = \frac{\lambda-1}{2}\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{12}^{(0)}\left[2\widetilde{\Sigma}_{22}^{(0)} - \widetilde{\Sigma}_{11}^{(0)}\right],$$

$$c_{12}^{(1)} = 3(\lambda-1)\left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-3}\widetilde{S}_{12}^{(0)}\widetilde{\Sigma}_{12}^{(0)} + \left(\widetilde{\overline{\Sigma}}^{(0)}\right)^{\lambda-1},$$

$$h_1 = \lambda\left(\widetilde{\widetilde{\Sigma}}^{(0)}\right)^{\lambda-1}\widetilde{S}_{11}^{(0)}D^{(1)},$$

$$h_2 = \lambda\left(\widetilde{\widetilde{\Sigma}}^{(0)}\right)^{\lambda-1}\widetilde{S}_{22}^{(0)}D^{(1)},$$

$$h_3 = \lambda\left(\widetilde{\widetilde{\Sigma}}^{(0)}\right)^{\lambda-1}\widetilde{S}_{12}^{(0)}D^{(1)},$$

$$m_1 = -\left(\cos^2\theta c_{11}^{(1)} + \sin^2\theta\ c_{11}^{(2)} + \sin 2\theta\ c_{11}^{(3)}\right),$$

$$m_2 = -\left(\cos^2\theta\ c_{22}^{(1)} + \sin^2\theta\ c_{22}^{(2)} + \sin 2\theta\ c_{22}^{(3)}\right),$$

$$m_3 = -\left(\cos^2\theta\ c_{12}^{(1)} + \sin^2\ \theta c_{12}^{(2)} + \sin 2\theta\ c_{12}^{(3)}\right)$$

with $s_1 = s_0((n/(1+q)) + \beta + 3) + 1$. The second-order time-dependent amplitudes $A^{(1)}(t)$ and $G^{(1)}(t)$ are

$$G^{(1)}(t) = k_2^{(1)}\left[G^{(0)}(t)\right]^{(n/(1+q))+\beta+3} \quad \text{and} \quad A^{(1)}(t) = \alpha^{-1}k_1^{(1)}G^{(1)}(t),$$

where $k_1^{(1)}$ and $k_2^{(1)}$, like $k_1^{(0)}$ and $k_2^{(0)}$, are two constants to be determined.

In Eqs. (89)–(93) the two constants, $k_1^{(0)}$ and $k_2^{(0)}$, were taken as unity to simplify these expressions.

The corresponding boundary conditions are

$$\widetilde{U}_2^{(1)}(0) = 0, \quad \widetilde{\Sigma}_{12}^{(1)}(0) = 0, \quad \widetilde{\Sigma}_{12}^{(1)}(\pi) = 0, \quad \widetilde{\Sigma}_{22}^{(1)}(\pi) = 0, \quad \text{and} \quad \varphi_3^{(1)}(0) = 0. \tag{99}$$

4. Discussions and numeric results

As shown in Eq. (44), the radial variation of stress is primarily dependent on the first-order stress exponent s_0. Since the viscoplastic creep exponent n is a positive quantity (typically, $n = 4$ to 6), s_0 is always negative for $q = 0$ according to Eq. (54). Also, s_0 remains less than zero for a primary creeping solid provided $q < n - 1$, which covers many engineering materials. For tertiary creeping ($q < 0$) solids, s_0 can be negative or positive, depending on the value of q. When $q < 0$ but $|q| < 1$, we have $s_0 < 0$. And if $|q| > 1$ with $q < 0$, $s_0 > 0$. Obviously, since the analysis is based on the condition (30) that requires $s_0 < 0$, the results obtained are not applicable to the case with $q < 0$ while $|q| > 1$.

The first-order time-dependent amplitudes, $A^{(0)}(t)$ and $G^{(0)}(t)$, are explicitly obtained from Eqs. (70) and (71). It is interesting to note that when $q = 0$, $A^{(0)}(t)$ and $G^{(0)}(t)$ are not functions of time but only dependent on the crack tip velocity $\dot{a}$, the material constants, and the constants, $k_1^{(0)}$ and $k_2^{(0)}$. For the case of $q \neq 0$, it is easy to show that

$$\left\{A^{(0)}(t), G^{(0)}(t)\right\} \propto t^{q/(n-q-1)}$$

with $G^{(0)}(0) = 0$. Take $q = 2$, then

$$\left\{A^{(0)}(t), G^{(0)}(t)\right\} \propto t^{2/(n-3)}.$$

Correspondingly,

$$\dot{\varepsilon}_{ij} \propto t^{2/(n-3)} \quad \text{and} \quad \varepsilon_{ij} \propto t^{(n-1)/(n-3)}.$$

This differs from Andrade's formulation (Kanninen and Popelar, 1985), $\varepsilon \propto t^{1/3}$, for uniaxial primary creep with $q = 2$, since in the present analysis the dynamic effect is included.

The constants, $k_1^{(0)}$, $k_2^{(0)}$, $k_1^{(1)}$ and $k_2^{(1)}$ remain undetermined. The computational results illustrated below are for $k_1^{(0)} = k_2^{(0)} = k_1^{(1)} = k_2^{(1)} = 1$. It can be verified that these parameters, within a rather large range, are not sensitive to the angular distributions of stresses and velocities.

The δ^0 order asymptotic equations described by Eqs. (73)–(77), together with the boundary conditions (83)–(85), pose a two-point boundary value problem. We can use either the shooting method or the relaxation method to solve the problem.

Fig. 2 shows the angular variations of stresses $\widetilde{\Sigma}_{ij}^{(0)}(\theta)$ $(i, j = 1, 2)$, equivalent stress $\widetilde{\widetilde{\Sigma}}^{(0)}(\theta)$, and hydrostatic stress $\widetilde{\Sigma}_{kk}^{(0)}(\theta)$, with $n/(1 + q) = 4$, $N = 0.25$ and $v = 0.3$. Comparison of this figure with Figs. 1–3 in Part 1, we can see that the angular variations of stresses for the two cases are quite different. Here, the stresses, $\widetilde{\Sigma}_{11}^{(0)}(\theta)$ and $\widetilde{\widetilde{\Sigma}}^{(0)}(\theta)$, no longer vary monotonically with respect to θ. Also, for $\widetilde{\Sigma}_{22}^{(0)}(\theta)$, the maximum value is not at $\theta = 0$ but is approximately located at $\pi/4$.

Fig. 3 gives the angular variations of velocities $\widetilde{U}_i^{(0)}(\theta)$ $(i = 1, 2)$ corresponding to the stress distributions shown in Fig. 2. It is shown that the variation of the absolute value of the velocity $\widetilde{U}_1^{(0)}(\theta)$ is not monotonic. That is, $|\widetilde{U}_1^{(0)}(\theta)|$ first decreases and then increases with θ reaching a minimum at $\theta = \pi$. The variation of $\widetilde{U}_2^{(0)}(\theta)$ is also not monotonic. In particular, the maximal value of $|\widetilde{U}_2^{(0)}(\theta)|$ is not at $\theta = \pi$ but approximately at $\theta = 3\pi/4$.

The angular variations of stresses and velocities for the δ^1 order asymptotic approximation are described by Eqs. (89)–(93). We can see that this is another two-point boundary value problem with the boundary conditions described by Eq. (99). Since these equations are a system of non-homogeneous but linear ordinary differential equations, closed form analytical solutions can be obtained. However, note that these equations contain the δ^0 order solution that can only be obtained numerically. Hence, only formal integrals could be given in the final form of solutions. Thus, a numerical scheme is more practical to solve Eqs. (89)–(93).

Fig. 4 shows the angular variations of stresses $\widetilde{\Sigma}_{ij}^{(1)}(\theta)$ $(i, j = 1, 2)$, with the material constants being the same as shown in Fig. 2.

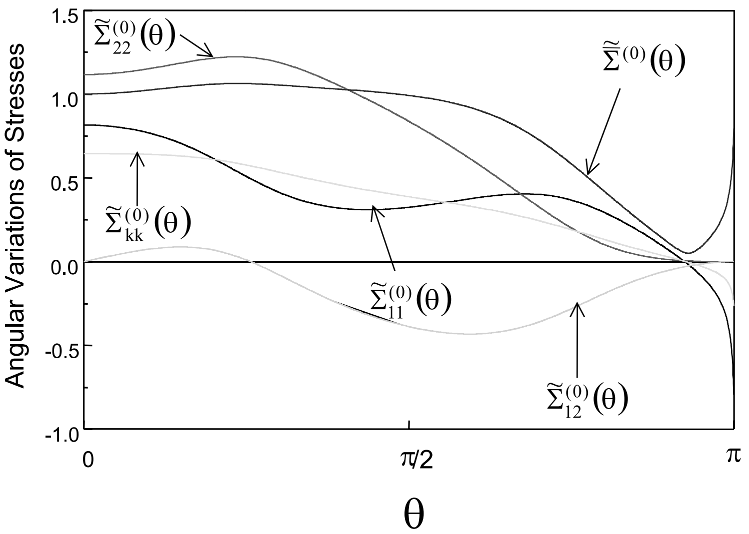

Fig. 2. Angular variations of the stresses $\widetilde{\Sigma}_{ij}^{(0)}(\theta)$ $(i, j = 1, 2)$, equivalent stress $\widetilde{\widetilde{\Sigma}}^{(0)}(\theta)$, and hydrostatic stress $\widetilde{\Sigma}_{kk}^{(0)}(\theta)$, with $n/(1 + q) = 4$, $N = 0.25$ and $v = 0.3$.

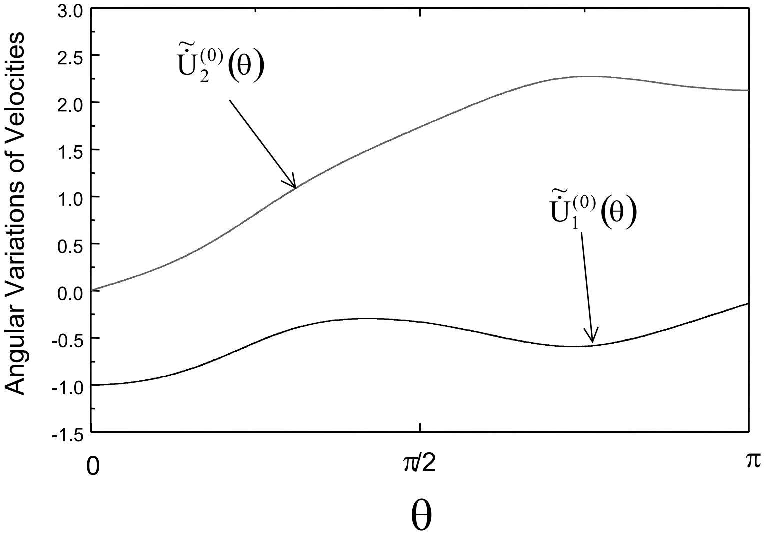

Fig. 3. Angular variations of velocities $\widetilde{\dot{U}}_i^{(0)}(\theta)$ $(i = 1, 2)$ corresponding to the stress distribution shown in Fig. 2.

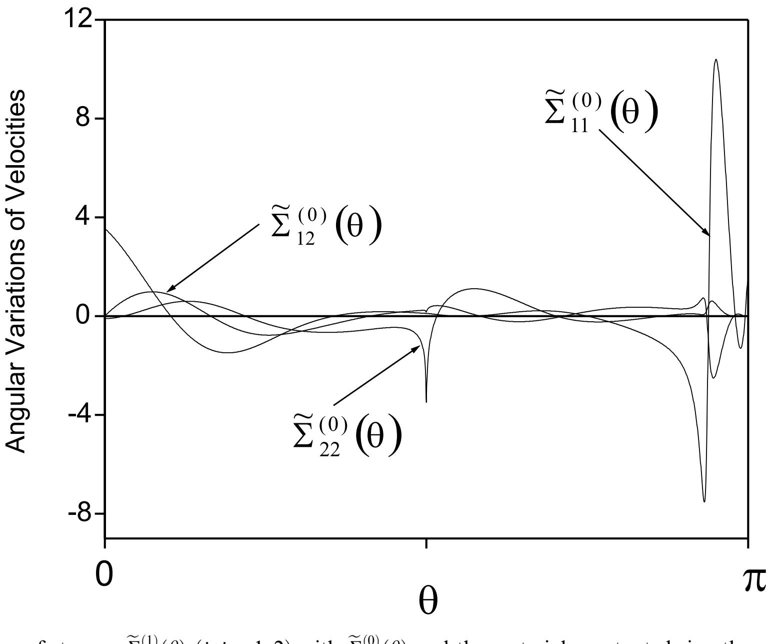

Fig. 4. Angular variations of stresses $\widetilde{\Sigma}_{ij}^{(1)}(\theta)$ $(i, j = 1, 2)$ with $\widetilde{\Sigma}_{ij}^{(0)}(\theta)$ and the material constants being the same as shown in Fig. 2.

Fig. 5 gives the angular variations of velocities $\widetilde{\dot{U}}_i^{(1)}(\theta)$ $(i = 1, 2)$ corresponding to the stress distributions shown in Fig. 4. Further discussion is also available to study the influence of material constants, such as N and n, on the second order tip behaviour. However, since $\widetilde{\Sigma}_{ij}^{(1)}(\theta)$ and $\widetilde{\dot{U}}_i^{(1)}(\theta)$ are perturbations of $\widetilde{\Sigma}_{ij}^{(0)}(\theta)$ and $\widetilde{\dot{U}}_i^{(0)}(\theta)$ with δ, respectively, the crack tip behaviour in the regime concerned are primarily characterised by $\widetilde{\Sigma}_{ij}^{(0)}(\theta)$ and $\widetilde{\dot{U}}_i^{(0)}(\theta)$ when δ is far smaller than unity. Therefore, we will not give much attention to it.

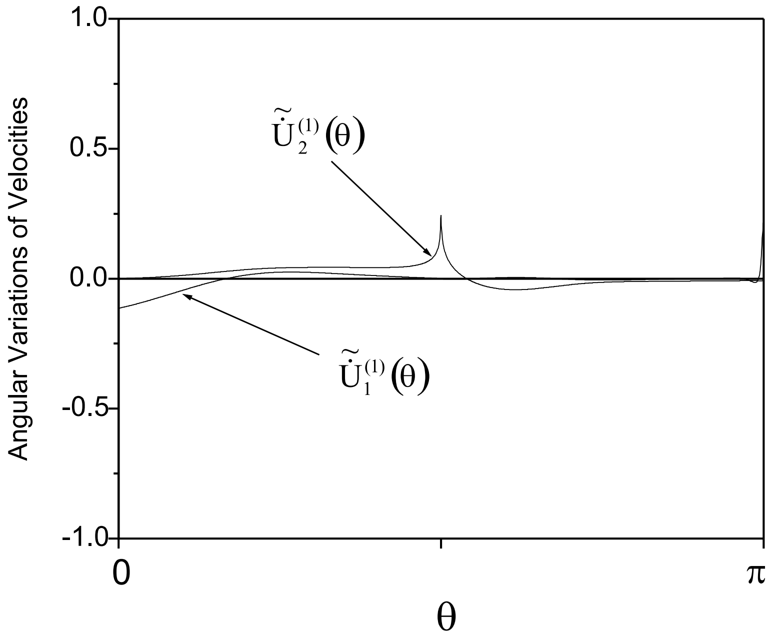

Fig. 5. Angular variations of velocities $\tilde{\dot{U}}_i^{(1)}(\theta)$ $(i = 1, 2)$ corresponding to the stress distribution shown in Fig. 4.

In the above figures only the first and second order asymptotic solutions are presented. When the characteristic damage is sufficiently small, say, $\delta < 0.1$, these results are adequate to describe the crack tip fields. Remember that $D = 1$ is an ideal mathematical limit corresponding to material rupture. D usually has a value between 0.2 and 0.5 even for the most ductile metals. Therefore, $\delta \sim 0.1$ indicates rather severe damage in practice. Thus, the present study is of practical importance.

In addition, all the numerical results given in this Part of the paper are based on the condition that the elastic effect is negligible. Discussions for which the elastic effect cannot be ignored are carried out in another study (Lu et al., 2001b).

5. Concluding remarks

In Part 1 and the present Part, comprehensive studies on two significant cases of fast fracture were carried out with different mathematical treatments. The asymptotic solutions obtained in both studies apply to different regions in the vicinity of the crack tip. We may again use the schematic to show the difference (see Fig. 1). Region A is the zone where damage dominates the field. In this region stresses vary with $r^{(1+N)/(1+[1+n/(1+q)]N)}$ when $r \to 0$. Region C is the regime where damage behaves as a perturbation. Within this regime, stresses vary according to $R^{-(n/(1+q)-1)^{-1}}$ for $R \geqslant R^*$, where R^* is determined by the loading and the material constants. It should be pointed out that although we do not specify an upper bound for R^*, R^* must be confined to a certain scale within which the solutions expressed by Eqs. (43) and (44) are adequate to describe the crack tip field. Obviously, in remote regions, Eqs. (43) and (44) are no more valid and other forms of solutions have to be provided. Region B in Fig. 1 is a regime where the damage is at a median level and it may play an equivalent role with stress concentration in the stress field. Since our analyses are based on the condition that the damage variable D is close to either unity or zero, the first and second order asymptotic solutions we offered are not appropriate to describe the behaviour in Region B. Even if the stress field of Region B can be approximately described with a sufficient number of higher order asymptotic solutions, such a methodology is usually not efficient. Alternatively, it is recommended that a fully numerical treatment is needed to tackle the problem. As Region B is relatively away from the crack tip, its geometry does not greatly influence the numerical computations.

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